

SPECTRAL VISCOSITY METHOD WITH GENERALIZED HERMITE FUNCTIONS FOR NONLINEAR CONSERVATION LAWS

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ABSTRACT. In this paper, we propose new spectral viscosity methods based on the generalized Hermite functions for the solution of nonlinear scalar conservation laws in the whole line. It is shown rigorously that these schemes converge to the unique entropy solution by using compensated compactness arguments, under some conditions. The numerical experiments of the inviscid Burger's equation support our result, and it verifies the reasonableness of the conditions.

1. INTRODUCTION

The spectral methods [9] approximate the exact solution of partial differential equations by seeking an “good” projection in the linear subspace spanned by various orthogonal systems of special functions. The resulting spectral accuracy is highly preferred than any other numerical method, especially when the solution is known to be globally smooth enough. Therefore, they are very appropriate for the elliptic and parabolic equations, thanks to the regularization properties of the operators. When mentioning the nonlinear conservation laws, it is well known that the solution may develop spontaneous jump discontinuity, i.e., shock waves. This irregularity of the solution destroys not only the accuracy of the spectral approximations at the point of discontinuity, but also that in the entire computational domain. It causes the oscillations throughout the domain, which is the so-called Gibb's phenomenon. Moreover, the instability is induced in the nonlinear case. It is shown in [25] that the usual spectral approximate solution may not converge to the entropy solution, the physically relevant one.

Despite all these deficiencies, many mathematicians still pay their effort to deal with these problems. The problems caused by the irregularity has already been solved for piecewise smooth functions in bounded domain or periodic piecewise smooth function in unbounded domain by filter techniques or reconstruction methods such as the Gegenbauer partial sum, see details in a series of papers [12], [11], [10], [28] and references therein. And the instability of the usual spectral approximations can be avoided by introducing the vanishing viscosity, which was first established by E. Tadmor [24]. The main idea of the spectral viscosity method is the use of artificial diffusion to stabilize the spectral computation without sacrificing its spectral accuracy. The periodic spectral method has been further investigated in [19], [25] and [20], etc. The nonperiodic Legendre spectral viscosity method is first introduced by Y. Maday, et. al. [18]. And H. Ma proposed the nonperiodic Chebyshev-Legendre spectral viscosity method in [16], [17]. For more literatures related to the spectral viscosity methods with various orthogonal basis in bounded domain, we refer the readers to [6], [8], [13] and references therein.

As we know, a large amount of physical problems are modeled in unbounded domain. During the past two decades, more and more attentions were paid to numerical solutions of differential equations defined in unbounded domains. Among the existing literature, the Hermite and Laguerre spectral methods are the most commonly used approaches based on orthogonal polynomials in infinite interval, referring to [7], [29]. Although the Hermite polynomials appear to be a natural choice of orthogonal basis of $L^2(\mathbb{R})$, it is not as popular as Fourier series and Chebyshev polynomials, due to its poor resolution (see [9]) and the lack of the analogue of fast Fourier transformation (FFT), see [4]. However, it is shown in [2] that the poor resolution can be remedied by a suitable choice

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of scaling factor. Some further investigations on the scaling factor can be found in [26] and also in Chapter 7, [22]. Recently, a guideline of choosing the suitable scaling factors for Gaussian/super-Gaussian functions is summarized by the author and S. S.-T. Yau in [14], where the Hermite spectral method is used to resolve the conditional density function of the states in nonlinear filtering problems.

The literatures on the spectral method in unbounded domains have already been not as rich as those in bounded domains, let alone the spectral viscosity method in unbounded domains. As far as we know, J. Aguirre and J. Rivas [1] is the only paper that considered the spectral viscosity method based on the Hermite functions. However, they defined the Hermite functions in the weighted $L_w^2(\mathbb{R})$, where $w(x) = e^{-x^2}$. And no scaling factor is introduced. This essentially causes their involved theoretical proof of the convergence rate of their proposed scheme. And it is more costly when they try to implement their scheme numerically.

In this paper, we shall revisit the nonlinear scalar conservation laws in \mathbb{R} :

$$(1.1) \quad \begin{cases} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, & x \in \mathbb{R}, t > 0 \\ u(x, 0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $f \in C^1$ is a smooth nonlinear function and $u_0 \in L^\infty(\mathbb{R})$. In general, the spontaneous jump discontinuity may be developed. Therefore, we can't expect the classical solutions to this problem. Moreover, we restrict ourselves to the physically relevant weak solution, the entropy solution, by imposing the entropy condition

$$(1.2) \quad \frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} \leq 0$$

in the sense of distributions, for all entropy pairs (U, F) , with $U \in C^2$ convex and $F'(u) = U'(u)f'(u)$, see [21].

We propose Hermite spectral viscosity methods based on generalized Hermite functions with two different viscosity terms.

(I) with viscosity term $\epsilon \partial_x \mathcal{D}_x u$:

$$(1.3) \quad \begin{cases} \partial_t u_N + \partial_x (P_{N+1} f(u_N)) - \epsilon_N \partial_x \mathcal{D}_x Q_{m_N} u_N = 0, & x \in \mathbb{R}, t \in (0, T), \\ u_N(x, 0) = P_N u_0(x), & x \in \mathbb{R}, \end{cases}$$

where ϵ_N is a positive parameter depending on N tending to 0 as N tends to ∞ , and Q_{m_N} is a viscosity operator which modifies only the high modes of the Fourier-Hermite expansion, that is,

$$(1.4) \quad Q_{m_N} \left(\sum_{k=0}^N \hat{\phi}_k(t) H_k^\alpha(x) \right) = \sum_{k=0}^N \hat{q}_k \hat{\phi}_k(t) H_k^\alpha(x),$$

with

$$(1.5) \quad \begin{cases} \hat{q}_k = 0, & \text{if } k \leq m_N \\ 1 - \frac{m_N}{k} \leq \hat{q}_k < 1, & \text{if } k > m_N \end{cases},$$

and $m_N < N$ is a positive integer that tends to ∞ with N .

(II) with viscosity term $\epsilon \mathcal{L}_\alpha u$:

$$(1.6) \quad \begin{cases} \partial_t v_N + \partial_x (P_{N+1} f(v_N)) + \epsilon_N \mathcal{L}_\alpha v_N = 0, & x \in \mathbb{R}, t \in (0, T), \\ v_N(x, 0) = P_N u_0(x), & x \in \mathbb{R}, \end{cases}$$

where \mathcal{L}_α is defined in (2.6), ϵ_N is a positive parameter depending on N tending to 0 as N tends to ∞ .

The convergences of both schemes have been rigorously shown under certain conditions.

(I) with viscosity term $\epsilon \partial_x \mathcal{D}_x u$:

Theorem 1.1. *Let $f \in C^1(\mathbb{R})$ be a nonlinear function such that $f(0) = 0$, and there exists a primitive function \bar{F} of $x f'(x)$, i.e., $\bar{F}' = x f'(x)$. Assume further that $u_0 \in L^2(\mathbb{R})$. Let u_N be the solution to the spectral approximation (1.3), which is uniformly bounded, i.e. $\|u_N\|_\infty < C$, independent of N , where $\|\circ\|_\infty = \sup_{\mathbb{R} \times (0,T)} |\circ|$ and the assumption*

$$(1.7) \quad \|xu_N\|_{L^2(0,T;L^2(\mathbb{R}))} \lesssim N^\theta,$$

for some $0 < \theta < \frac{1}{4}$, holds. Let $N^{-\frac{1}{2}} \ll \epsilon_N \ll N^{-2\theta}$, $m_N \ll N^\beta$, with some $0 < \beta < \theta$. Then $\{u_N\}$ converges (strongly in $L^p_{loc}(\Omega)$, $1 \leq p < \infty$) to the unique entropy solution, u , of the problem (1.1), where $\Omega \in \mathbb{R} \times [0, T]$ is an open and bounded subset.

(II) with viscosity term $\epsilon \mathcal{L}_\alpha u$:

Theorem 1.2. *Let $f \in C^1(\mathbb{R})$ be a nonlinear function such that $f(0) = 0$, and there exists a primitive function \bar{F} of $x f'(x)$. Assume further that $u_0 \in L^2(\mathbb{R})$. Let v_N be the solution to the spectral approximation (1.6), which is uniformly bounded. We assume further that*

$$(1.8) \quad \|x^2 v_N\|_{L^1(\mathbb{R} \times (0,T))} \ll \frac{1}{\epsilon_N}.$$

Let $\frac{1}{\epsilon_N \sqrt{N}} \rightarrow 0$. Then $\{v_N\}$ converges (strongly in $L^p_{loc}(\Omega)$, $1 \leq p < \infty$) to the unique entropy solution, u , of the problem (1.1), where $\Omega \in \mathbb{R} \times [0, T]$ is an open and bounded subset.

Compared to the scheme in [1], the schemes in our paper have at least two advantages:

- The viscosity term $\epsilon \mathcal{L}_\alpha u$ stabilized the numerical scheme, due to its symmetry and positivity, while the stability of the scheme in [1] and the scheme with viscosity term $\epsilon \partial_x \mathcal{D}_x u$ in our paper are not guaranteed;
- Our schemes can be implemented easily with the help of the scaling factor. The better resolution and fewer oscillations are retained with much smaller truncation terms N even without viscosity, see Figure 4.1, compared with Fig 6.1-6.3, [1].

We provide two alternative schemes to solve the nonlinear conservation laws in \mathbb{R} . The convergences are guaranteed under some reasonable conditions (3.10) or (3.36). It is hard to tell whether these conditions are weaker or not than the one in [1], i.e., $\|xu_N\|_\infty < C$, independent of N , due to the unbounded domain $\mathbb{R} \times (0, T)$.

The paper is organized as follows. In section 2, we give the definition of the generalized Hermite functions and their properties. The new Hermite spectral viscosity methods are proposed in section 3. The convergences of our schemes have been rigorously shown under some conditions by using the compensated compactness arguments. The reasonableness of the conditions have been verified numerically in section 4 for the inviscid Burger's equation.

2. GENERALIZED HERMITE FUNCTIONS

In this section, we introduce the generalized Hermite functions and derive some properties which are inherited from the Hermite polynomials. To establish the convergence rate of the Hermite spectral method, we shall also state the convergence rate of the orthogonal approximation.

Let $L^2(\mathbb{R})$ be the Lebesgue space, equipped with the norm $\|\cdot\| = (\int_{\mathbb{R}} |\cdot|^2 dx)^{\frac{1}{2}}$ and the scalar product $\langle \cdot, \cdot \rangle$. In the sequel, we shall follow the convection in the asymptotic analysis, $a \sim b$ means that there exists some generic constants $C_1, C_2 > 0$ such that $C_1 a \leq b \leq C_2 a$; $a \lesssim b$ means that there exists some generic constant $C_3 > 0$ such that $a \leq C_3 b$.

Let $\mathcal{H}_n(x)$ be the physical Hermite polynomials, i.e., $\mathcal{H}_n(x) = (-1)^n e^{x^2} \partial_x^n e^{-x^2}$, $n \geq 0$. The three-term recurrence

$$(2.1) \quad \mathcal{H}_0 \equiv 1, \quad \mathcal{H}_1(x) = 2x \quad \text{and} \quad \mathcal{H}_{n+1}(x) = 2x\mathcal{H}_n(x) - 2n\mathcal{H}_{n-1}(x).$$

is handier in implementation. One of the well-known and useful fact of Hermite polynomials is that they are mutually orthogonal with respect to the weight $w(x) = e^{-x^2}$. We define the generalized

Hermite functions with the scaling factor $\alpha > 0$ as

$$(2.2) \quad H_n^\alpha(x) = \left(\frac{\alpha}{2^n n! \sqrt{\pi}} \right)^{\frac{1}{2}} \mathcal{H}_n(\alpha x) e^{-\frac{1}{2}\alpha^2 x^2},$$

for $n \geq 0$. It is readily to derive the following properties for the generalized Hermite functions (2.2):

■ The $\{H_n^\alpha(x)\}_{n \in \mathbb{Z}^+}$ forms an orthonormal basis of $L^2(\mathbb{R})$, i.e.

$$(2.3) \quad \int_{\mathbb{R}} H_n^\alpha(x) H_m^\alpha(x) dx = \delta_{nm},$$

where δ_{nm} is the Kronecker function.

■ $H_n^\alpha(x)$ is the n^{th} eigenfunction of the following Sturm-Liouville problem

$$(2.4) \quad e^{\frac{1}{2}\alpha^2 x^2} \partial_x (e^{-\alpha^2 x^2} \partial_x (e^{\frac{1}{2}\alpha^2 x^2} u(x))) + \lambda_n u(x) = 0,$$

with the corresponding eigenvalue

$$(2.5) \quad \lambda_n = 2\alpha^2 n.$$

For conciseness, let us denote the Sturm-Liouville operator as

$$(2.6) \quad \mathcal{L}_\alpha(\circ) = -e^{\frac{1}{2}\alpha^2 x^2} \partial_x (e^{-\alpha^2 x^2} \partial_x (e^{\frac{1}{2}\alpha^2 x^2} \circ)).$$

■ By convention, $H_n^\alpha \equiv 0$, for $n < 0$. For $n \geq 0$, the three-term recurrence is inherited from the Hermite polynomials:

$$(2.7) \quad x H_n^\alpha(x) = \frac{\sqrt{\lambda_{n+1}}}{2\alpha^2} H_{n+1}^\alpha(x) + \frac{\sqrt{\lambda_n}}{2\alpha^2} H_{n-1}^\alpha(x).$$

■ The derivative of $H_n^\alpha(x)$ with respect to x

$$(2.8) \quad \partial_x H_n^\alpha(x) = -\frac{\sqrt{\lambda_{n+1}}}{2} H_{n+1}^\alpha(x) + \frac{\sqrt{\lambda_n}}{2} H_{n-1}^\alpha(x).$$

For convenience, let $\mathcal{D}_x = \partial_x + \alpha^2 x$. Then

$$(2.9) \quad \mathcal{D}_x H_n^\alpha(x) = \sqrt{2\alpha^2 n} H_{n-1}^\alpha(x) = \sqrt{\lambda_n} H_{n-1}^\alpha(x).$$

■ The “orthogonality” of $\{\mathcal{D}_x H_n^\alpha(x)\}_{n \in \mathbb{Z}^+}$ follows immediately from (2.3), i.e.,

$$(2.10) \quad \int_{\mathbb{R}} \mathcal{D}_x H_n^\alpha(x) \mathcal{D}_x H_m^\alpha(x) dx = 2\alpha^2 n \delta_{nm} = \lambda_n \delta_{nm},$$

where δ_{nm} is the Kronecker function.

Any function $u(x) \in L^2(\mathbb{R})$ can be written in the form

$$(2.11) \quad u(x) = \sum_{n=0}^{\infty} \hat{u}_n H_n^\alpha(x),$$

with

$$(2.12) \quad \hat{u}_n = \int_{\mathbb{R}} u(x) H_n^\alpha(x) dx.$$

where $\{\hat{u}_n\}_{n=0}^{\infty}$ are the Fourier-Hermite coefficients.

Let us denote the linear subspace of $L^2(\mathbb{R})$ spanned by the first N Hermite functions by

$$(2.13) \quad \mathcal{R}_N := \text{span}\{H_0^\alpha(x), \dots, H_N^\alpha(x)\}.$$

Remark 2.1. *Actually, we have the norms $\|\partial_x \phi\|$ controlled by $\|\mathcal{D}_x \phi\|$ and $\|\phi\|$, for any $\phi \in \mathcal{R}_N$. Let us consider*

$$\begin{aligned} \|\partial_x \phi\|^2 &= \left\| \sum_{k=0}^N \hat{\phi}_k \left(-\frac{\sqrt{\lambda_{k+1}}}{2} H_{k+1}^\alpha + \frac{\sqrt{\lambda_k}}{2} H_{k-1}^\alpha \right) \right\|^2 \\ &= \frac{1}{4} \sum_{k=0}^N \left\{ \hat{\phi}_k^2 \lambda_{k+1} - \hat{\phi}_{k+2} \hat{\phi}_k \sqrt{\lambda_{k+2} \lambda_{k+1}} - \hat{\phi}_k \hat{\phi}_{k-2} \sqrt{\lambda_k \lambda_{k-1}} + \hat{\phi}_k^2 \lambda_k \right\}, \end{aligned}$$

where $\hat{\phi}_k = 0$, for all $k > N$ or $k < 0$. Recall that

$$\left| \hat{\phi}_{k+2} \hat{\phi}_k \sqrt{\lambda_{k+2} \lambda_{k+1}} \right| \leq \frac{1}{2} (\hat{\phi}_{k+2}^2 \lambda_{k+2} + \hat{\phi}_k^2 \lambda_{k+1}).$$

We arrive at

$$(2.14) \quad \|\partial_x \phi\|^2 \lesssim \sum_{k=0}^N \hat{\phi}_k^2 \lambda_k + \alpha^2 \sum_{k=0}^N \hat{\phi}_k^2 = \|\mathcal{D}_x \phi\|^2 + \alpha^2 \|\phi\|^2.$$

Similarly, we can get

$$(2.15) \quad \|x\phi\|^2 \lesssim \frac{1}{\alpha^4} [\|\mathcal{D}_x \phi\|^2 + \alpha^2 \|\phi\|^2].$$

We define the L^2 -orthogonal projection $P_N^\alpha : L^2(\mathbb{R}) \rightarrow \mathcal{R}_N$, given $v \in L^2(\mathbb{R})$,

$$\langle v - P_N^\alpha v, \phi \rangle = 0,$$

for all $\phi \in \mathcal{R}_N$. More precisely,

$$P_N^\alpha v(x) = \sum_{n=0}^N \hat{v}_n H_n^\alpha(x),$$

where \hat{v}_n , $n = 0, \dots, N$, are the Fourier-Hermite coefficients defined in (2.11).

The error estimate of the orthogonal projection onto \mathcal{R}_N is readily shown in Theorem 4.2, [23] for $\alpha = 1$ and it can be trivially extended for $\alpha > 0$ that

Lemma 2.1. *For any $\mathcal{D}_x^m u \in L^2(\mathbb{R})$ with $m \geq 0$,*

$$\|\mathcal{D}_x^l(u - P_N^\alpha u)\| \lesssim \alpha^{l-m} N^{\frac{l-m}{2}} \|\mathcal{D}_x^m u\|, \quad 0 \leq l \leq m.$$

In the sequel, the superscript α in P_N^α will be dropped if no confusion will arise.

3. THE HERMITE SPECTRAL VISCOSITY METHOD

It is well known that the entropy solution to (1.1) can be obtained as the limit when the artificial introduced viscosity term vanishes. In this section, we shall introduce two appropriate viscosity terms $\epsilon \partial_x \mathcal{D}_x u$ and $\epsilon \mathcal{L}_\alpha u$. With the viscosity term $\epsilon \partial_x \mathcal{D}_x u$, the viscosity operator Q_{m_N} has been introduced in the spectral scheme as in [1], where only the high frequency terms appear in the artificial viscosity. The convergence of this scheme has been shown under the condition

$$(1.7) \quad \|xu_N\|_{L^2(0,T;L^2(\mathbb{R}))} \ll N^\theta,$$

for some $\theta \geq 0$ such that $N^\theta \ll \frac{1}{\epsilon_N}$ or $N^\theta \sim \frac{1}{\epsilon_N}$. The reasonableness of the condition (3.10) has been verified in Table 1 and the arguments in section 4.

3.1. With viscosity term $\epsilon \partial_x \mathcal{D}_x u$. The spectral scheme is introduced in (1.3). Let us first understand the viscosity operator Q_{m_N} (defined in (1.5)).

Lemma 3.2. *Let Q_{m_N} be defined as in (1.4), (1.5). Then*

$$(3.1) \quad \|\mathcal{D}_x \phi\|^2 \lesssim \|\mathcal{D}_x Q_{m_N} \phi\|^2 + \alpha^2 m_N^2 \|\phi\|^2,$$

and

$$(3.2) \quad \|\mathcal{D}_x Q_{m_N} \phi\|^2 \lesssim \|\mathcal{D}_x \phi\|^2 + \alpha^2 m_N^2 \|\phi\|^2.$$

for all $\phi \in \mathcal{R}_N$.

Proof. Let us show (3.1) in detail only, and (3.2) can be obtained by the similar argument. Let $\phi = \sum_{k=0}^N \hat{\phi}_k H_k^\alpha(x)$, and $R_{m_N} = I - Q_{m_N}$, where I is the identity operator. Then

$$(3.3) \quad \|\mathcal{D}_x \phi\|^2 \lesssim \|\mathcal{D}_x Q_{m_N} \phi\|^2 + \|\mathcal{D}_x R_{m_N} \phi\|^2.$$

We split ϕ in dyadic parts $\phi(x) = \sum_{k=0}^{m_N} \hat{\phi}_k H_k^\alpha(x) + \sum_{j=1}^J \phi^j(x)$, where

$$\phi^j(x) = \sum_{k=2^{j-1}m_N+1}^{2^j m_N} \hat{\phi}_k H_k^\alpha(x),$$

$j = 1, \dots, J$. Here $J = \log_2 \left(\frac{N}{m_N} \right) + 1$ and $\hat{\phi}_k = 0$ for $k = N+1, \dots, 2^J m_N$. From the orthogonality relation (2.10), one has

$$(3.4) \quad \|\mathcal{D}_x R_{m_N} \phi\|^2 = \left\| \mathcal{D}_x R_{m_N} \sum_{k=0}^{m_N} \hat{\phi}_k H_k^\alpha \right\|^2 + \sum_{j=1}^J \|\mathcal{D}_x R_{m_N} \phi^j\|^2.$$

We bound each summand above by using the fact that given R a linear operator defined in \mathcal{R}_N such that

$$R \left(\sum_{k=0}^N \hat{\phi}_k H_k^\alpha(x) \right) = \sum_{k=0}^N \hat{r}_k \hat{\phi}_k H_k^\alpha(x),$$

where $\hat{r}_0, \dots, \hat{r}_N$ are real numbers. Then for all $\phi \in \mathcal{R}_N$,

$$(3.5) \quad \|\mathcal{D}_x R \phi\|^2 \stackrel{(2.10)}{=} \sum_{k=0}^N \hat{r}_k^2 \hat{\phi}_k^2 \lambda_k \leq \left(\sum_{k=0}^N \hat{r}_k^2 \lambda_k \right) \left(\sum_{k=0}^N \hat{\phi}_k^2 \right) = \left(\sum_{k=0}^N \hat{r}_k^2 \lambda_k \right) \|\phi\|^2.$$

Since $\hat{q}_k = 0$, for $k \leq m_N$,

$$(3.6) \quad \left\| \mathcal{D}_x R_{m_N} \sum_{k=0}^{m_N} \hat{\phi}_k H_k^\alpha \right\|^2 \stackrel{(3.5)}{\leq} \left(\sum_{k=0}^{m_N} (1 - \hat{q}_k)^2 \lambda_k \right) \left(\sum_{k=0}^{m_N} \hat{\phi}_k^2 \right) \lesssim \alpha^2 m_N^2 \left(\sum_{k=0}^{m_N} \hat{\phi}_k^2 \right) = \alpha^2 m_N^2 \left\| \sum_{k=0}^{m_N} \hat{\phi}_k H_k^\alpha \right\|^2.$$

For the second summand on the right-hand side of (3.4), since $\hat{q}_k \geq 1 - \frac{m_N}{k}$, we have, for any $j = 1, \dots, J$,

$$(3.7) \quad \|\mathcal{D}_x R_{m_N} \phi^j\|^2 \stackrel{(3.5)}{\lesssim} \left(\sum_{k=2^{j-1}m_N+1}^{2^j m_N} (1 - \hat{q}_k)^2 \lambda_k \right) \|\phi^j\|^2 \lesssim \alpha^2 m_N^2 \sum_{k=2^{j-1}m_N+1}^{2^j m_N} \frac{1}{k} \|\phi^j\|^2 \lesssim \alpha^2 m_N^2 \|\phi^j\|^2.$$

Combine (3.6) and (3.7), we obtain that

$$(3.8) \quad \|\mathcal{D}_x R_{m_N} \phi\|^2 \lesssim \alpha^2 m_N^2 \left(\left\| \sum_{k=0}^{m_N} \hat{\phi}_k^2 H_k^\alpha \right\|^2 + \sum_{j=1}^J \|\phi^j\|^2 \right) \lesssim \alpha^2 m_N^2 \|\phi\|^2.$$

Substituting the above equation back to (3.3), we get the desired result (3.1). And (3.2) can be obtained similarly, with the fact that

$$\|\mathcal{D}_x Q_{m_N} \phi\|^2 \lesssim \|\mathcal{D}_x \phi\|^2 + \|\mathcal{D}_x R_{m_N} \phi\|^2 \stackrel{(3.8)}{\lesssim} \|\mathcal{D}_x \phi\|^2 + \alpha^2 m_N^2 \|\phi\|^2.$$

□

The apriori estimates on the approximate solution u_N are obtained in the following lemma.

Lemma 3.3. *Let $f \in C^1(\mathbb{R})$, and there exists a primitive function $\bar{F}(x)$ of $xf'(x)$, i.e. $\bar{F}'(x) = xf'(x)$. Let $u_0 \in L^2(\mathbb{R})$, $T > 0$, Q_{m_N} is given in (1.4), (1.5), and $u_N : [0, T] \times \mathbb{R} \rightarrow \mathcal{R}_N$ the solution of (1.3). Let us assume that there is a positive constant C , independent of N , such that*

$$(3.9) \quad \|u_N\|_\infty \leq C.$$

Furthermore, we assume that

$$(3.10) \quad \|xu_N\|_{L^2(0,T;L^2(\mathbb{R}))} \lesssim N^\theta,$$

for some $\theta > 0$. Then

$$(3.11) \quad \|\mathcal{D}_x Q_{m_N} u_N\|_{L^2(0,T;L^2(\mathbb{R}))} \lesssim \begin{cases} \frac{1}{\sqrt{\epsilon_N}}, & \text{if } \frac{1}{\sqrt{\epsilon_N}} \gg N^\theta \\ N^\theta, & \text{if } \frac{1}{\sqrt{\epsilon_N}} \ll N^\theta \end{cases},$$

and

$$(3.12) \quad \|u_N\|(t) \lesssim \begin{cases} 1, & \text{if } \frac{1}{\sqrt{\epsilon_N}} \gg N^\theta \\ \sqrt{\epsilon_N} N^\theta, & \text{if } \frac{1}{\sqrt{\epsilon_N}} \ll N^\theta \end{cases},$$

where the generic constant contains in \lesssim may depend on α, T etc., but not N , the norms are defined as $\|\circ\|_\infty = \|\circ\|_{L^\infty(\mathbb{R} \times (0, T))} = \sup_{\mathbb{R} \times [0, T]} |\circ|$ and $\|\circ\|_{L^2(0, T; L^2(\mathbb{R}))}^2 := \int_0^T \|\circ\|^2 dt$.

Proof. Let us choose $\varphi = u_N \in \mathcal{R}_N$ in (1.3) and it yields that

$$(3.13) \quad 0 \stackrel{(2.6)}{=} \int_{\mathbb{R}} u_N \partial_t u_N dx + \int_{\mathbb{R}} \partial_x (P_{N+1} f(u_N)) u_N dx - \epsilon_N \int_{\mathbb{R}} \partial_x \mathcal{D}_x (Q_{m_N} u_N) u_N dx.$$

It is clear that the first term on the right-hand side of (3.13) is $\frac{1}{2} \frac{d}{dt} \|u_N\|^2$ and the second term is zero. In fact, the second term gives

$$(3.14) \quad \begin{aligned} \int_{\mathbb{R}} \partial_x (P_{N+1} f(u_N)) u_N dx &= \int_{\mathbb{R}} P_{N+1} \partial_x (f(u_N)) u_N dx \\ &\quad - \frac{1}{2} \sqrt{\lambda_{N+2}} \int_{\mathbb{R}} \left[\widehat{f(u_N)}_{N+1}(t) H_{N+2}^\alpha(x) + \widehat{f(u_N)}_{N+2}(t) H_{N+1}^\alpha(x) \right] u_N dx \\ &= \int_{\mathbb{R}} P_{N+1} \partial_x (f(u_N)) u_N dx = \int_{\mathbb{R}} \partial_x (f(u_N)) u_N dx = \int_{\mathbb{R}} f'(u_N) u_N \partial_x u_N dx, \end{aligned}$$

where the first equality of (3.14) follows from the fact that

$$(3.15) \quad P_N \partial_x \phi(x, t) - \partial_x P_N \phi(x, t) = \frac{1}{2} \sqrt{\lambda_{N+1}} \left[\hat{\phi}_N(t) H_{N+1}^\alpha(x) + \hat{\phi}_{N+1}(t) H_N^\alpha(x) \right],$$

the second and third equalities of (3.14) hold due to the orthogonality of generalized Hermite function. If there exists a primitive function $\bar{F}(x)$ of $xf'(x)$, with the fact that for any N , $\lim_{|x| \rightarrow \pm\infty} u_N(x) = 0$, we obtain that

$$\int_{\mathbb{R}} f'(u_N) u_N du_N = \bar{F}(u_N(x)) \Big|_{x=\pm\infty} = 0.$$

Next, we shall examine the last term on the right-hand side of (3.13). Using integration by parts, we have

$$(3.16) \quad \begin{aligned} -\epsilon_N \int_{\mathbb{R}} \partial_x \mathcal{D}_x Q_{m_N} u_N u_N dx &= \epsilon_N \int_{\mathbb{R}} \mathcal{D}_x Q_{m_N} u_N \partial_x u_N dx \\ &= \epsilon_N \int_{\mathbb{R}} \mathcal{D}_x Q_{m_N} u_N \mathcal{D}_x u_N dx - \epsilon_N \alpha^2 \int_{\mathbb{R}} \mathcal{D}_x Q_{m_N} u_N (xu_N) dx \\ &= I - \epsilon_N \alpha^2 II. \end{aligned}$$

Let us compute I and II term by term:

$$(3.17) \quad \begin{aligned} I &\stackrel{(2.9)}{=} \epsilon_N \int_{\mathbb{R}} \left(\sum_{k=0}^N \hat{q}_k \hat{u}_k \sqrt{\lambda_k} H_{k-1}^\alpha \right) \left(\sum_{m=0}^N \hat{u}_m \sqrt{\lambda_m} H_{m-1}^\alpha \right) dx \stackrel{(2.3)}{=} \epsilon_N \sum_{k=0}^N \hat{q}_k \hat{u}_k^2 \lambda_k \\ &\stackrel{(2.10)}{>} \epsilon_N \sum_{k=0}^N \hat{q}_k^2 \hat{u}_k^2 \lambda_k = \epsilon_N \int_{\mathbb{R}} |\mathcal{D}_x Q_{m_N} u_N|^2 dx = \epsilon_N \|\mathcal{D}_x Q_{m_N} u_N\|^2, \end{aligned}$$

since $\hat{q}_k < 1$, and II can be estimated as

$$II \leq \frac{1}{2\gamma} \|\mathcal{D}_x Q_{m_N} u_N\|^2 + \frac{\gamma}{2} \|xu_N\|^2,$$

with $\gamma > \frac{\alpha^2}{2}$, by Young's inequality. Therefore, equation (3.13) can be estimated as

$$(3.18) \quad 0 \geq \frac{1}{2} \frac{d}{dt} \|u_N\|^2 + \epsilon_N \left(1 - \frac{\alpha^2}{2\gamma} \right) \|\mathcal{D}_x Q_{m_N} u_N\|^2 - \frac{\epsilon_N \alpha^2 \gamma}{2} \|xu_N\|^2.$$

Integrating from 0 to T , we get

$$\epsilon_N \alpha^2 \gamma \|xu_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2 + \|u_0\|^2 \geq \|u_N\|^2(T) + 2\epsilon_N \left(1 - \frac{\alpha^2}{2\gamma}\right) \|\mathcal{D}_x Q_{m_N} u_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2.$$

Hence, (3.11) and (3.12) are followed immediately. \square

We are now ready to show Theorem 1.1.

The proof of Theorem 1.1. The uniform boundedness of $\{u_N\}$ in $L^\infty(\mathbb{R} \times [0, T])$ guarantees that there exists a subsequence converges in the weak-* sense of L^∞ , denote this subsequence also $\{u_N\}$ and the limit u . We shall prove that u is the unique entropy solution of (1.1), and the whole sequence $\{u_N\}$ tends to u in $L_{loc}^p(\Omega)$, $1 \leq p < \infty$.

We first show that $\partial_t u_N + \partial_x f(u_N)$ is in a compact set of $H_{loc}^{-1}(\mathbb{R} \times (0, T))$.

$$(3.19) \quad \partial_t u_N + \partial_x f(u_N) = \epsilon_N \partial_x \mathcal{D}_x Q_{m_N} u_N + \partial_x [(I - P_{N+1})f(u_N)].$$

Let K be a compact set of $\mathbb{R} \times (0, T)$. It is obvious that the first term on the right-hand side of (3.19) tends to 0 in $H_{loc}^{-1}(\mathbb{R} \times (0, T))$, since

$$(3.20) \quad \epsilon_N \|\mathcal{D}_x Q_{m_N} u_N\|_{L^2(K)} \stackrel{(3.11)}{\lesssim} \epsilon_N \frac{1}{\sqrt{\epsilon_N}} \rightarrow 0.$$

According to Lemma 2.1, the second term on the right-hand side of (3.19) can be estimated as

$$(3.21) \quad \|(I - P_{N+1})f(u_N)\|_{L^2(K)} \lesssim N^{-\frac{1}{2}} \|\mathcal{D}_x f(u_N)\|_{L^2(0,T;L^2(\mathbb{R}))},$$

Notice that

$$(3.22) \quad \|\mathcal{D}_x f(u_N)\| \leq \|\partial_x f(u_N)\| + \alpha^2 \|xf(u_N)\| \leq \sup_{|\xi| \leq \|u_N\|_\infty} |f'(\xi)| (\|\partial_x u_N\| + \alpha^2 \|xu_N\|),$$

where we use the fact that there exists ξ , such that $|\xi| \leq \|u_N\|_\infty$ and $f(u_N) = f'(\xi)u_N$, if $f(0) = 0$ and $f \in C^1(\mathbb{R})$. By Remark 2.1 and Lemma 3.2, we have

$$(3.23) \quad \|\partial_x u_N\| \leq \|\mathcal{D}_x u_N\| + \alpha \|u_N\| \leq \|\mathcal{D}_x Q_{m_N} u_N\| + \alpha(m_N + 1)\|u_N\|.$$

Thus, back to (3.21), we obtain that

$$(3.24) \quad \|(I - P_{N+1})f(u_N)\|_{L^2(K)} \lesssim N^{-\frac{1}{2}} \left(\frac{1}{\sqrt{\epsilon_N}} + m_N + N^\theta \right) \ll \frac{1}{\sqrt{\epsilon_N N}} + N^{-\frac{1}{2}} m_N \rightarrow 0,$$

since $N^{-\frac{1}{2}} \ll \epsilon_N \ll N^{-2\theta}$ and $m_N \ll N^\beta$, with $0 < \beta < \theta < \frac{1}{4}$. Therefore, we conclude that $\partial_t u_N + \partial_x f(u_N)$ is in a compact set of $H_{loc}^{-1}(\mathbb{R} \times (0, T))$.

Let (U, F) be an entropy pair associated to (1.1). Next, we shall show that $\partial_t U(u_N) + \partial_x F(u_N)$ is also in a compact subset of $H_{loc}^{-1}(\mathbb{R} \times (0, T))$.

$$(3.25) \quad \begin{aligned} \partial_t U(u_N) + \partial_x F(u_N) &= U'(u_N)(\partial_t u_N + \partial_x f(u_N)) \\ &= \epsilon_N U'(u_N) \partial_x \mathcal{D}_x Q_{m_N} u_N + U'(u_N) \partial_x (I - P_{N+1})f(u_N) \\ &= \epsilon_N \partial_x (U'(u_N) \mathcal{D}_x Q_{m_N} u_N) - \epsilon_N U''(u_N) \partial_x u_N \mathcal{D}_x Q_{m_N} u_N \\ &\quad + \partial_x (U'(u_N)(I - P_{N+1})f(u_N)) - U''(u_N) \partial_x u_N (I - P_{N+1})f(u_N). \end{aligned}$$

The first and third term on the right-hand side of (3.25) can be estimated similarly as in (3.20) and (3.24). Indeed,

$$\epsilon_N \|U'(u_N) \mathcal{D}_x Q_{m_N} u_N\|_{L^2(K)} \leq \epsilon_N \|U'(u_N)\|_\infty \|\mathcal{D}_x Q_{m_N} u_N\|_{L^2(0,T;L^2(\mathbb{R}))} \stackrel{(3.11)}{\lesssim} \epsilon_N \frac{1}{\sqrt{\epsilon_N}} \rightarrow 0,$$

and

$$\begin{aligned} \|U'(u_N)(I - P_{N+1})f(u_N)\|_{L^2(K)} &\leq \|U'(u_N)\|_\infty \|(I - P_{N+1})f(u_N)\|_{L^2(0,T;L^2(\mathbb{R}))} \\ &\stackrel{(3.24)}{\ll} \frac{1}{\sqrt{\epsilon_N N}} + N^{-\frac{1}{2}} m_N \rightarrow 0. \end{aligned}$$

Therefore, $\epsilon_N \partial_x(U'(u_N) \mathcal{D}_x Q_{m_N} u_N) \rightarrow 0$ and $\partial_x(U'(u_N)(I - P_{N+1})f(u_N)) \rightarrow 0$ in $H_{loc}^{-1}(\mathbb{R} \times (0, T))$. The second and fourth term of the right-hand side of (3.25) are estimated as

$$\begin{aligned}
& \epsilon_N \|U''(u_N) \partial_x u_N \mathcal{D}_x Q_{m_N} u_N\|_{L^1(K)} \\
& \leq \epsilon_N \|U''(u_N)\|_{\infty} \|\partial_x u_N\|_{L^2(0,T;L^2(\mathbb{R}))} \|\mathcal{D}_x Q_{m_N} u_N\|_{L^2(0,T;L^2(\mathbb{R}))} \\
& \stackrel{(3.23)}{\lesssim} \epsilon_N \|\mathcal{D}_x Q_{m_N} u_N\|_{L^2(0,T;L^2(\mathbb{R}))} (\|\mathcal{D}_x Q_{m_N} u_N\|_{L^2(0,T;L^2(\mathbb{R}))} + m_N \|u_N\|_{L^2(0,T;L^2(\mathbb{R}))}) \\
& \stackrel{(3.11),(3.12)}{\lesssim} \epsilon_N \frac{1}{\sqrt{\epsilon_N}} \left(\frac{1}{\sqrt{\epsilon_N}} + m_N \right) = \mathcal{O}(1),
\end{aligned}$$

since $\sqrt{\epsilon_N} m_N \ll N^{-\theta+\beta} \rightarrow 0$, as $N \rightarrow \infty$. And

$$\begin{aligned}
& \|U''(u_N) \partial_x u_N (I - P_{N+1}) f(u_N)\|_{L^1(K)} \\
& \leq \|U''(u_N)\|_{\infty} \|\partial_x u_N\|_{L^2(0,T;L^2(\mathbb{R}))} \|(I - P_{N+1}) f(u_N)\|_{L^2(0,T;L^2(\mathbb{R}))} \\
(3.26) \quad & \stackrel{(3.23),(3.24)}{\ll} \left(\frac{1}{\sqrt{\epsilon_N}} + m_N \right) N^{-\frac{1}{2}} \left(\frac{1}{\sqrt{\epsilon_N}} + m_N \right) \leq \frac{1}{\epsilon_N \sqrt{N}} + \frac{m_N^2}{\sqrt{N}} \rightarrow 0,
\end{aligned}$$

since $\frac{1}{\epsilon_N \sqrt{N}} \ll \frac{1}{N^{-\frac{1}{2}} \sqrt{N}} = 1$ and $\frac{m_N^2}{\sqrt{N}} \ll N^{2\beta-\frac{1}{2}} \rightarrow 0$, with the assumption that $\beta < \theta < \frac{1}{4}$.

Thus the entropy production $\partial_t U(u_N) + \partial_x F(u_N)$ can be written as a sum of four terms, two are bounded in $L^1(\Omega)$ and the other two tend to 0 in $H_{loc}^{-1}(\Omega)$. Besides, $\partial_t U(u_N) + \partial_x F(u_N)$ is in $W_{loc}^{-1,p}(\mathbb{R} \times (0, T))$ for any $p > 2$, since U and F are continuous and u_N is uniformly bounded in $L^\infty(\mathbb{R} \times (0, T))$. Therefore, in view of the Murat's lemma [5], $\partial_t U(u_N) + \partial_x F(u_N)$ is in a compact subset of $H_{loc}^{-1}(\mathbb{R} \times (0, T))$.

We conclude that the entropy production of (1.3) is H^{-1} -compact, by compensated compactness arguments [27], implies that u_N converges strongly in $L_{loc}^p(\Omega)$, $1 \leq p < \infty$ to a weak solution, u , of the conservation law (1.1).

It remains to show that u is indeed the weak solution of (1.1) satisfying the entropy condition. Let us multiply a nonnegative test function $\phi \in C_0^1(\mathbb{R} \times (0, T))$ on both sides of (3.25) and integrate it with respect to both t and x :

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}} [\partial_t U(u_N) + \partial_x F(u_N)] \phi dx dt \\
& \stackrel{(3.25)}{=} -\epsilon_N \int_0^T \int_{\mathbb{R}} U'(u_N) \mathcal{D}_x Q_{m_N} u_N \partial_x \phi dx dt - \epsilon_N \int_0^T \int_{\mathbb{R}} U''(u_N) \partial_x u_N \mathcal{D}_x Q_{m_N} u_N \phi dx dt \\
(3.27) \quad & - \int_0^T \int_{\mathbb{R}} U'(u_N) (I - P_{N+1}) f(u_N) \partial_x \phi dx dt - \int_0^T \int_{\mathbb{R}} U''(u_N) \partial_x u_N (I - P_{N+1}) f(u_N) \phi dx dt.
\end{aligned}$$

The first, third and fourth term on the right-hand side of (3.27) tend to 0, as $N \rightarrow \infty$. In fact,

$$\begin{aligned}
(3.28) \quad & \epsilon_N \left| \int_0^T \int_{\mathbb{R}} U'(u_N) \mathcal{D}_x Q_{m_N} u_N \partial_x \phi dx dt \right| \\
& \leq \epsilon_N \|U'(u_N)\|_{\infty} \|\mathcal{D}_x Q_{m_N} u_N\|_{L^2(0,T;L^2(\mathbb{R}))} \|\partial_x \phi\|_{L^2(0,T;L^2(\mathbb{R}))} \lesssim \epsilon_N \frac{1}{\sqrt{\epsilon_N}} \rightarrow 0; \\
& \left| \int_0^T \int_{\mathbb{R}} U'(u_N) (I - P_{N+1}) f(u_N) \partial_x \phi dx dt \right| \\
& \leq \|U'(u_N)\|_{\infty} \|(I - P_{N+1}) f(u_N)\|_{L^2(0,T;L^2(\mathbb{R}))} \|\partial_x \phi\|_{L^2(0,T;L^2(\mathbb{R}))} \\
(3.29) \quad & \stackrel{(3.24)}{\ll} \frac{1}{\sqrt{\epsilon_N N}} + N^{-\frac{1}{2}} m_N \rightarrow 0;
\end{aligned}$$

and

$$\begin{aligned}
& \left| \int_0^T \int_{\mathbb{R}} U''(u_N) \partial_x u_N (I - P_{N+1}) f(u_N) \phi dx dt \right| \\
& \leq \|U''(u_N)\|_{\infty} \|\partial_x u_N\|_{L^2(0,T;L^2(\mathbb{R}))} \|(I - P_{N+1}) f(u_N)\|_{L^2(0,T;L^2(\mathbb{R}))} \|\phi\|_{\infty} \\
(3.30) \quad & \stackrel{(3.26)}{\lesssim} \frac{1}{\epsilon_N \sqrt{N}} + \frac{m_N^2}{\sqrt{N}} \rightarrow 0.
\end{aligned}$$

We shall analyze the third term on the right-side of (3.27). Notice that $Q_{m_N} = I - R_{m_N}$, then

$$\begin{aligned}
& -\epsilon_N \int_0^T \int_{\mathbb{R}} U''(u_N) \partial_x u_N \mathcal{D}_x Q_{m_N} u_N \phi dx dt \\
& = -\epsilon_N \int_0^T \int_{\mathbb{R}} U''(u_N) (\mathcal{D}_x u_N)^2 \phi dx dt + \epsilon_N \int_0^T \int_{\mathbb{R}} U''(u_N) \mathcal{D}_x u_N \mathcal{D}_x R_{m_N} u_N \phi dx dt \\
(3.31) \quad & + \epsilon_N \alpha^2 \int_0^T \int_{\mathbb{R}} U''(u_N) x u_N \mathcal{D}_x Q_{m_N} u_N \phi dx dt.
\end{aligned}$$

The second and third term on the right-hand side of (3.31) tend to 0, as $N \rightarrow \infty$. In fact,

$$\begin{aligned}
& \epsilon_N \left| \int_0^T \int_{\mathbb{R}} U''(u_N) \mathcal{D}_x u_N \mathcal{D}_x R_{m_N} u_N \phi dx dt \right| \\
& \leq \epsilon_N \|U''(u_N)\|_{\infty} \|\mathcal{D}_x u_N\|_{L^2(0,T;L^2(\mathbb{R}))} \|\mathcal{D}_x R_{m_N} u_N\|_{L^2(0,T;L^2(\mathbb{R}))} \|\phi\|_{\infty} \\
& \stackrel{(3.1),(3.8)}{\lesssim} \epsilon_N \left(\frac{1}{\sqrt{\epsilon_N}} + m_N \right) m_N \rightarrow 0,
\end{aligned}$$

and

$$\begin{aligned}
& \epsilon_N \alpha^2 \left| \int_0^T \int_{\mathbb{R}} U''(u_N) x u_N \mathcal{D}_x Q_{m_N} u_N \phi dx dt \right| \\
& \leq \epsilon_N \alpha^2 \|U''(u_N)\|_{\infty} \|x u_N\|_{L^2(0,T;L^2(\mathbb{R}))} \|\mathcal{D}_x Q_{m_N} u_N\|_{L^2(0,T;L^2(\mathbb{R}))} \|\phi\|_{\infty} \lesssim \epsilon_N N^{\theta} \frac{1}{\sqrt{\epsilon_N}} \rightarrow 0.
\end{aligned}$$

And due to the convexity of U , the first term on the right-hand side of (3.31) is nonpositive. Therefore, as $N \rightarrow \infty$, the second term on the right-hand side of (3.27) is nonpositive. Combining (3.28)-(3.30), we conclude that for any nonnegative test function $\phi \in C_0^1(\mathbb{R} \times (0, T))$,

$$\lim_{N \rightarrow \infty} \int_0^T \int_{\mathbb{R}} [\partial_t U(u_N) + \partial_x F(u_N)] \phi dx dt \leq 0.$$

That is, the entropy condition (1.2) has been satisfied in the weak sense. \square

Remark 3.2. The conditions on ϵ_N and m_N are almost the same as those in [1]. The difference is that we replace the condition $\|x u_N\|_{\infty} < C$, independent of N , by some growth condition on $\|x u_N\|_{L^2(0,T;L^2(\mathbb{R}))}$ ((3.10) with $\theta < \frac{1}{4}$). It is hard to tell which condition is more restrictive.

3.2. With viscosity term $\epsilon \mathcal{L}_{\alpha} u$. In the next subsection, we introduce another viscosity term $\epsilon \mathcal{L}_{\alpha} u$. Unlike the viscosity operator Q_{m_N} only modified the high frequency terms, this viscosity includes all. The spectral viscosity method is introduced in (1.6). Let us start with the apriori estimates on v_N .

Lemma 3.4. Let $f \in C^1(\mathbb{R})$, and there exists a primitive function $\bar{F}(x)$ of $x f'(x)$, i.e. $\bar{F}'(x) = x f'(x)$. Let $u_0 \in L^2(\mathbb{R})$, $T > 0$ and $v_N : [0, T] \times \mathbb{R} \rightarrow \mathcal{R}_N$ the solution of (1.6). Let us assume that there is a positive constant C , independent of N and α , such that

$$(3.32) \quad \|v_N\|_{\infty} \leq C.$$

Then

$$(3.33) \quad \|\mathcal{D}_x v_N\|_{L^2(0,T;L^2(\mathbb{R}))} \lesssim \frac{1}{\sqrt{\epsilon_N}},$$

and

$$(3.34) \quad \|v_N\|(t) \leq \|u_0\|.$$

Proof. We multiply (1.6) by $\varphi = v_N \in \mathcal{R}_N$ and integrate with respect to x :

$$(3.35) \quad 0 = \int_{\mathbb{R}} (\partial_t v_N) v_N dx + \int_{\mathbb{R}} \partial_x (f(v_N)) v_N dx + \epsilon_N \int_{\mathbb{R}} (\mathcal{L}_\alpha v_N) v_N dx = \frac{1}{2} \frac{d}{dt} \|v_N\|^2 + \epsilon_N \|\mathcal{D}_x v_N\|^2,$$

where the first and second term on the right-hand side of (3.35) are given by the same argument as in Lemma 3.3. The third term is followed from the fact that

$$\begin{aligned} \int_{\mathbb{R}} (\mathcal{L}_\alpha \psi) \phi dx &\stackrel{(2.6)}{=} - \int_{\mathbb{R}} e^{\frac{1}{2}\alpha^2 x^2} \partial_x (e^{-\alpha^2 x^2} \partial_x (e^{\frac{1}{2}\alpha^2 x^2} \psi)) \phi dx = - \int_{\mathbb{R}} ((\partial_x - \alpha^2 x) \mathcal{D}_x \psi) \phi dx \\ &= \int_{\mathbb{R}} \mathcal{D}_x \psi \mathcal{D}_x \phi dx. \end{aligned}$$

Integrating on both sides of (3.35) from 0 to T , we obtain that

$$\|u_0\|^2 \geq \|v_N\|^2(T) + 2\epsilon_N \|\mathcal{D}_x v_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2.$$

Equation (3.33) and (3.34) follow immediately. \square

Theorem 3.3. *Let $f \in C^1(\mathbb{R})$ be a nonlinear function such that $f(0) = 0$, and there exists a primitive function \bar{F} of $xf'(x)$. Assume further that $u_0 \in L^2(\mathbb{R})$. Let v_N be the solution to the spectral approximation (1.6), which is uniformly bounded, i.e. (3.32) holds. We assume further that*

$$(3.36) \quad \|x^2 v_N\|_{L^1(\mathbb{R} \times (0,T))} \ll \frac{1}{\epsilon_N}.$$

Let $\frac{1}{\epsilon_N \sqrt{N}} \rightarrow 0$. Then $\{v_N\}$ converges (strongly in $L_{loc}^p(\Omega)$, $1 \leq p < \infty$) to the unique entropy solution, u , of the problem (1.1), where $\Omega \in \mathbb{R} \times [0, T]$ is an open and bounded subset.

Proof. The uniform boundedness of $\{v_N\}$ in $L^\infty(\mathbb{R} \times [0, T])$ guarantees that there exists a subsequence converges in the weak-* sense of L^∞ , denote this subsequence also $\{v_N\}$ and the limit u . We shall prove that u is the unique entropy solution of (1.1), and the whole sequence $\{v_N\}$ tends to u in $L_{loc}^p(\Omega)$, $1 \leq p < \infty$.

We first show that $\partial_t u_N + \partial_x f(u_N)$ is in a compact set of $H_{loc}^{-1}(\mathbb{R} \times (0, T))$.

$$\begin{aligned} \partial_t v_N + \partial_x f(v_N) &= -\epsilon_N \mathcal{L}_\alpha v_N + \partial_x [(I - P_{N+1})f(v_N)] \\ &= \epsilon_N \partial_x \mathcal{D}_x v_N - \epsilon_N \alpha^2 \partial_x (x v_N) + \epsilon_N \alpha^2 v_N - \epsilon_N \alpha^4 x^2 v_N + \partial_x [(I - P_{N+1})f(v_N)] \\ (3.37) \quad &= \epsilon_N \partial_x^2 v_N + \epsilon_N \alpha^2 v_N - \epsilon_N \alpha^4 x^2 v_N + \partial_x [(I - P_{N+1})f(v_N)]. \end{aligned}$$

Let $K \subset \mathbb{R} \times (0, T)$ be a compact set. Notice that

$$\begin{aligned} \epsilon_N \|\partial_x v_N\|_{L^2(K)} &\stackrel{(2.14)}{\leq} \epsilon_N (\|\mathcal{D}_x v_N\|_{L^2(0,T;L^2(\mathbb{R}))} + \alpha \|v_N\|_{L^2(0,T;L^2(\mathbb{R}))}) \\ (3.38) \quad &\stackrel{(3.33),(3.34)}{\lesssim} \epsilon_N \frac{1}{\sqrt{\epsilon_N}} \rightarrow 0; \end{aligned}$$

$$(3.39) \quad \epsilon_N \|v_N\|_{L^2(K)} \lesssim \epsilon_N \rightarrow 0;$$

$$\begin{aligned} \|(I - P_{N+1})f(v_N)\|_{L^2(K)} &\lesssim N^{-\frac{1}{2}} \|\mathcal{D}_x f(v_N)\|_{L^2(0,T;L^2(\mathbb{R}))} \\ &\stackrel{(3.22)}{\lesssim} N^{-\frac{1}{2}} (\|\partial_x v_N\|_{L^2(0,T;L^2(\mathbb{R}))} + \alpha^2 \|x v_N\|_{L^2(0,T;L^2(\mathbb{R}))}) \\ (3.40) \quad &\stackrel{(2.14),(2.15)}{\lesssim} N^{-\frac{1}{2}} \frac{1}{\sqrt{\epsilon_N}} \rightarrow 0; \end{aligned}$$

and

$$(3.41) \quad \epsilon_N \|x^2 v_N\|_{L^1(K)} \ll \epsilon_N \frac{1}{\epsilon_N} = 1.$$

Thus, $\partial_t v_N + \partial_x f(v_N)$ can be written as a sum of four terms, two tend to 0 in $H_{loc}^{-1}(\mathbb{R})$, one is bounded in $L_{loc}^1(\mathbb{R})$ and the other one tends to 0 in $L_{loc}^2(\mathbb{R})$. Besides, $\partial_t v_N + \partial_x f(v_N)$ is in $W_{loc}^{-1,p}(\mathbb{R} \times (0, T))$

for any $p > 2$, since $f \in C^2$ and u_N is uniformly bounded in $L^\infty(\mathbb{R} \times (0, T))$. Therefore, in view of the Murat's lemma [5], $\partial_t v_N + \partial_x f(v_N)$ is in a compact subset of $H_{loc}^{-1}(\mathbb{R} \times (0, T))$.

Next, we show that $\partial_t U(v_N) + \partial_x F(v_N)$ is also in a compact subset of $H_{loc}^{-1}(\mathbb{R} \times (0, T))$, where (U, F) is the entropy pair introduced in (1.2).

$$\begin{aligned} \partial_t U(v_N) + \partial_x F(v_N) &= -\epsilon_N U'(v_N) \mathcal{L}_\alpha v_N + U'(v_N) \partial_x (I - P_{N+1}) f(v_N) \\ &\stackrel{(3.37)}{=} \epsilon_N \partial_x (U'(v_N) \partial_x v_N) - \epsilon_N U''(v_N) (\partial_x v_N)^2 + \epsilon_N U'(v_N) \alpha^2 v_N - \epsilon_N U'(v_N) \alpha^4 x^2 v_N \\ (3.42) \quad &+ \partial_x (U'(v_N) \partial_x (I - P_{N+1}) f(v_N)) - U''(v_N) \partial_x v_N (I - P_{N+1}) f(v_N). \end{aligned}$$

Notice the estimates in (3.38)-(3.41) and the fact that $U \in C^2$, $\|v_N\|_\infty < C$, the first, third, fourth and fifth term on the right-hand side of (3.42) can be handled similarly as before, i.e. the first and fifth term tend to 0 in $H_{loc}^{-1}(\mathbb{R} \times (0, T))$; the third term tends to 0 in $L_{loc}^2(\mathbb{R} \times (0, T))$; and the fourth term is bounded in $L_{loc}^1(\mathbb{R} \times (0, T))$. The rest two terms on the right-hand side of (3.42) are both bounded in $L_{loc}^1(\mathbb{R} \times (0, T))$. In fact,

$$\epsilon_N \|U''(v_N) (\partial_x v_N)^2\|_{L^1(K)} \leq \epsilon_N \|U''(v_N)\|_\infty \|\partial_x v_N\|_{L^2(0, T; L^2(\mathbb{R}))}^2 \lesssim \epsilon_N \left(\frac{1}{\sqrt{\epsilon_N}} \right)^2 = 1,$$

and

$$\begin{aligned} &\|U''(v_N) \partial_x v_N (I - P_{N+1}) f(v_N)\|_{L^1(K)} \\ &\leq \|U''(v_N)\|_\infty \|\partial_x v_N\|_{L^2(0, T; L^2(\mathbb{R}))} \|(I - P_{N+1}) f(v_N)\|_{L^2(0, T; L^2(\mathbb{R}))} \\ (3.43) \quad &\stackrel{(3.38), (3.40)}{\lesssim} \frac{1}{\sqrt{\epsilon_N}} N^{-\frac{1}{2}} \frac{1}{\sqrt{\epsilon_N}} = \frac{1}{\epsilon_N \sqrt{N}} \rightarrow 0. \end{aligned}$$

Therefore, as we argued before, in view of the Murat's lemma [5], $\partial_t U(v_N) + \partial_x F(v_N)$ is in a compact subset of $H_{loc}^{-1}(\mathbb{R} \times (0, T))$. We conclude that v_N converges strongly in $L_{loc}^p(\mathbb{R} \times (0, T))$, $1 \leq p < \infty$ to a weak solution, u , of the conservation law (1.1).

It remains to show that the entropy condition (1.2) is satisfied in the weak sense. Let us multiply a nonnegative test function $\phi \in C_0^1(\mathbb{R} \times (0, T))$ on both sides of (3.42) and integrate it with respect to both t and x :

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} [\partial_t U(v_N) + \partial_x F(v_N)] \phi dx dt \\ &\stackrel{(3.42)}{=} -\epsilon_N \int_0^T \int_{\mathbb{R}} U'(v_N) \partial_x v_N \partial_x \phi dx dt - \epsilon_N \int_0^T \int_{\mathbb{R}} U''(v_N) (\partial_x v_N)^2 \phi dx dt \\ &\quad + \epsilon_N \int_0^T \int_{\mathbb{R}} U'(v_N) \alpha^2 v_N \phi dx dt - \epsilon_N \int_0^T \int_{\mathbb{R}} U'(v_N) \alpha^4 x^2 v_N \phi dx dt \\ (3.44) \quad &- \int_0^T \int_{\mathbb{R}} U'(v_N) \partial_x (I - P_{N+1}) f(v_N) \partial_x \phi dx dt - \int_0^T \int_{\mathbb{R}} U''(v_N) \partial_x v_N (I - P_{N+1}) f(v_N) \phi dx dt. \end{aligned}$$

Estimates (3.38)-(3.41) and (3.43) imply that all the terms except the second one on the right-hand side of (3.44) tends to 0, as $N \rightarrow \infty$. Meanwhile, the second term is nonpositive, due to the convexity of U . Therefore, the entropy condition is satisfied in the weak sense, i.e. for any nonnegative test function $\phi \in C_0^1(\mathbb{R} \times (0, T))$,

$$\lim_{N \rightarrow \infty} \int_0^T \int_{\mathbb{R}} [\partial_t U(v_N) + \partial_x F(v_N)] \phi dx dt \leq 0.$$

□

Compare the viscosity term in (1.3) and (1.6), the convergence analysis for (1.6) is easier, but the price to pay is the condition (3.36) is stronger than (3.10), since

$$\|x v_N\|_{L^2(\mathbb{R} \times (0, T))}^2 \leq \|v_N\|_\infty \|x^2 v_N\|_{L^1(\mathbb{R} \times (0, T))} \ll \frac{1}{\epsilon_N}.$$

4. NUMERICAL EXPERIMENTS

In this section we use the spectral viscosity methods (1.3) or (1.6) to numerically solve the inviscid Burger's equation

$$(4.1) \quad \partial_t u + \frac{1}{2} \partial_x (u^2) = 0,$$

in \mathbb{R} , with the initial condition $u(x, 0) = e^{-x^2}$. We compute the same problem in [1] for the purpose of comparison. The exact solution is given implicitly by the method of characteristics, i.e.,

$$(4.2) \quad u(\eta + te^{-\eta^2}, t) = e^{-\eta^2},$$

with the initial condition $u(\eta, 0) = u_0$. And the shock presents at time $T^* = \left(\frac{e}{2}\right)^{\frac{1}{2}} \approx 1.1658$. All of the numerical results displayed below are at time $t = 1.5 > T^*$.

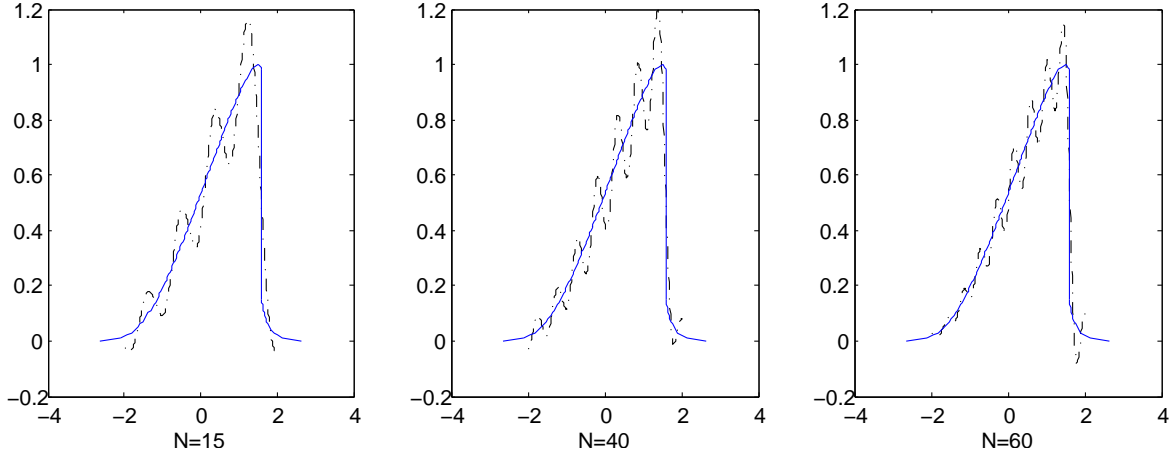


FIGURE 4.1 Solid blue line: the exact solution of Burger's equation; Dashed line: the spectral approximation without viscosity, with $N = 15$, $N = 40$ and $N = 60$, respectively.

In our numerical simulations, we implement the scheme (1.3) or (1.6) and overcome the difficulty of accurately approximating the Fourier-Hermite coefficients (mentioned in section 6 [1]) with the help of the suitable scaling factor α . The optimal choice of the scaling factor to accurately resolve the functions is still open, let alone the solution to some partial differential equations. But the suitable choice of the scaling factor is investigated in [26], [2], [3], [14], etc. It is known so far that the scaling factor should match the asymptotical behavior of the function/solution to be resolved. From (4.2), for any fixed time t , it is easy to see that $u(x, t) \sim e^{-x^2}$, as $|x| \gg 1$. According to the practical guidelines in [14], we choose $\alpha = \sqrt{2}$. Generally speaking, the asymptotical behavior of the solutions to evolution equations is time-dependent. A time-dependent scaling factor should improve the resolution further. The study on the time-dependent scaling factor of the Hermite spectral method in solving evolution equations has been recently investigated in [15]. Here, we focus ourselves on the intrinsic problem by using constant scaling factor.

In both spectral schemes, we let $\varphi = H_m^\alpha(x)$, $m = 0, 1, \dots, N$, in (1.3) or (1.6). The coefficients $\tilde{u}_m(t)$ or $\tilde{v}_m(t)$, $m = 0, \dots, N$, are the solution of a nonlinear system of ordinary differential equation. It is solved by using the fourth order Runge-Kutta method with an adaptive time step (*ode45* in Matlab).

In Figure 4.1, we show the result of the spectral approximation without any viscosity for $N = 15, 40$ and 60 , respectively. The larger N is, the better resolution at the point of discontinuity we achieve, but the oscillations do not disappear with the increase of N . The Gibb's phenomenon prevents the convergence of our method, even in the intervals where the exact solution is actually smooth. Compared with the pseudospectral viscosity method based on the Hermite functions introduced in [1] (Figure 6.1-6.3), the discontinuity can be resolved better with much smaller N .

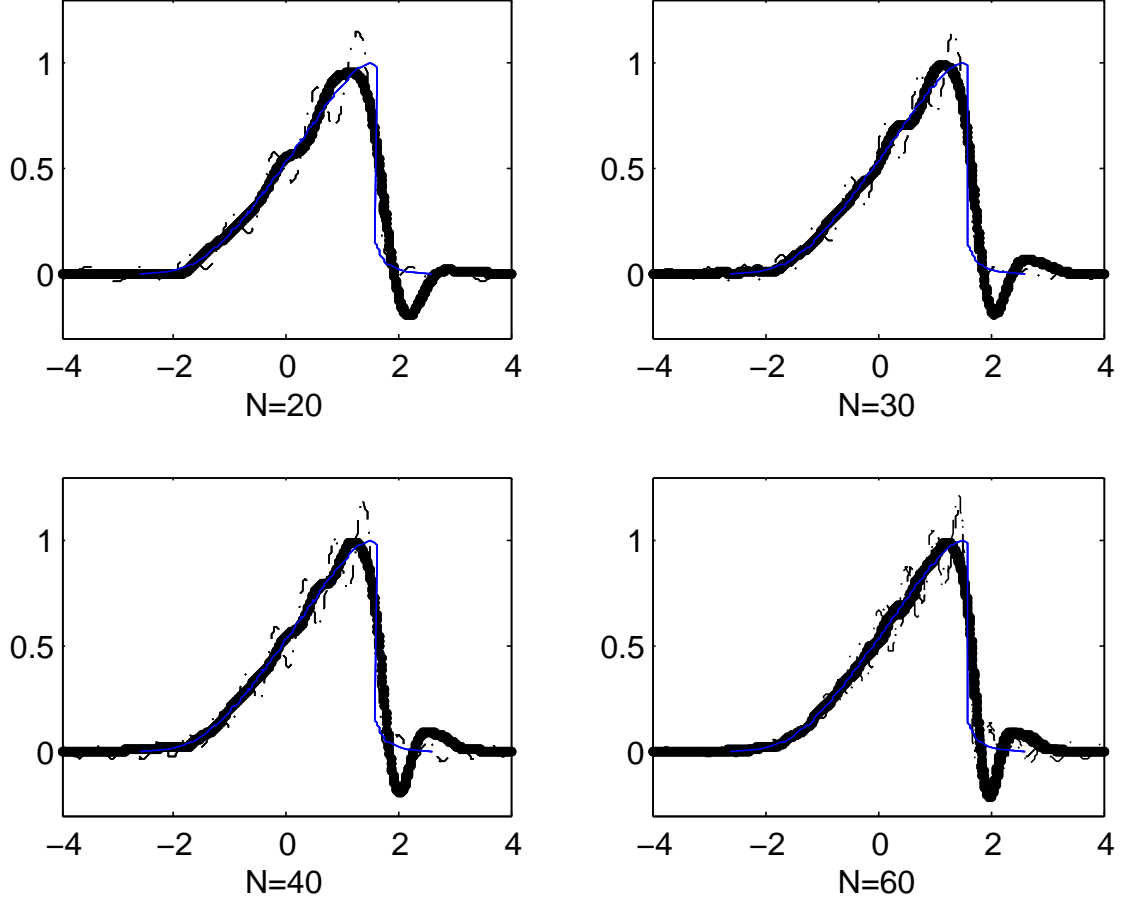


FIGURE 4.2 Solid blue line: the exact solution of inviscid Burger's equation; Dashed line: the spectral approximation without viscosity with $N = 20, 30, 40$ and 60 , respectively. Bold dotted line: the first three are produced by the spectral viscosity method (1.3) with $\epsilon_N = 0.5N^{-0.33}$, $m_N = 5N^{0.16}$, q_k defined in (1.5) and different N s.

N	40	45	50	55	60	65	70
$\ \mathcal{D}_x u_N\ _{L^2(0,T;L^2(\mathbb{R}))}^2$	3.0621	3.1076	3.1504	3.1926	3.2350	3.2766	3.3145
$\ xu_N\ _{L^2(0,T;L^2(\mathbb{R}))}^2$	0.9690	0.9682	0.9676	0.9671	0.9667	0.9665	0.9664
$\ u_N\ _{L^2(0,T;L^2(\mathbb{R}))}^2$	1.8756	1.8757	1.8758	1.8759	1.8760	1.8762	1.8763

TABLE 1 $\|\mathcal{D}_x u_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2$, $\|xu_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2$ and $\|u_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2$ versus N , respectively, are displayed, where u_N is the solution obtained by the spectral viscosity method (1.3) with $\epsilon_N = 0.5N^{-0.33}$.

The viscosity introduced in the spectral approximation (1.3) depends on the parameters ϵ_N , m_N and the operator Q_{m_N} . Let us try the following multipliers \hat{q}_k used in [1]:

$$(4.3) \quad \hat{q}_k = \frac{N}{N - m_N} \left(1 - \frac{m_N}{k} \right),$$

for $k > m_N$. It is easy to check that condition (1.5) are satisfied.

In Figure 4.2, we add the viscosity term in (1.3) by suitably tuning the parameters $\epsilon_N = 0.5N^{-0.33}$ and $m_N = 5N^{0.16}$. We plot the results at $t = 1.5$ with viscosity operator \hat{q}_k above and different $N = 20, 30, 40$ and 60 , respectively. The conditions on m_N, ϵ_N in Theorem 1.1 are satisfied. It is observed that the convergence is better than without viscosity. And it is clear to see that the oscillations do not alleviated as N increases; meanwhile, the discontinuity is resolved better with larger N . Table 1 is used to numerically verify the condition (3.10) and the apriori estimate on $\|\mathcal{D}_x Q_{m_N} u_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2$ (3.11). The norm $\|\circ\|_{L^2(\mathbb{R})}^2(t)$ at every time step is computed in the frequency side by Parseval's identity, and the integration in time in $\|\circ\|_{L^2(0,T;L^2(\mathbb{R}))}^2$ is performed by the trapezoid rule. The time steps are given by the adaptive algorithm in *ode45* in Matlab. The command “polyfit” in Matlab is used to find the minimal mean square linear fit of the growth rate of $\|\mathcal{D}_x u_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2$, $\|xu_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2$ and $\|u_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2$ with respect to N , which are $N^{0.1420}$, $N^{-0.0049}$ and $N^{0.0007}$, respectively. It numerically confirms that $\|xu_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2 \lesssim N^{-0.0049}$, which can be interpreted as bounded independent of N , i.e. condition (3.10) is satisfied. And $\|\mathcal{D}_x Q_{m_N} u_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2 \lesssim N^{0.1420} + m_N N^{0.0007} \lesssim N^{0.1420} + N^{2*0.16+0.0007} \ll \frac{1}{\epsilon_N} \approx N^{0.33}$, that is, the apriori estimate (3.11) is correct.

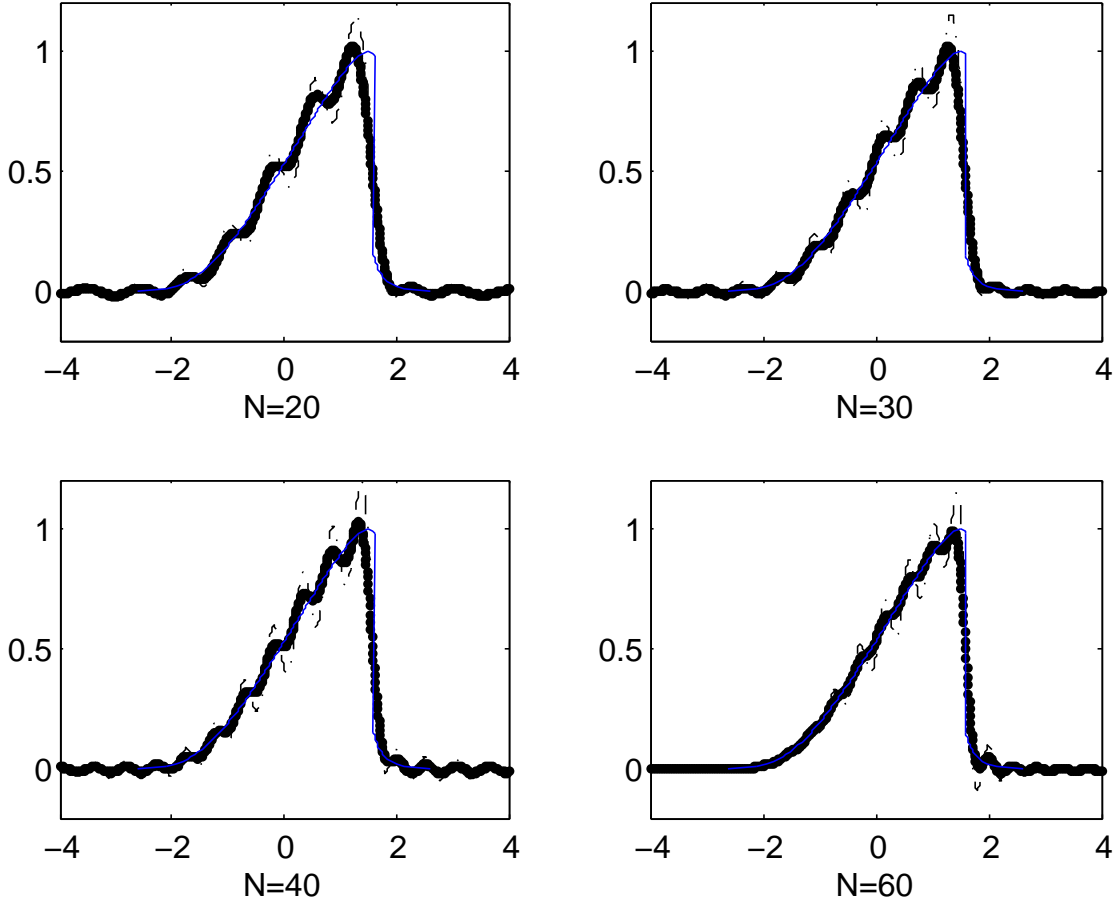


FIGURE 4.3 Solid blue line: the exact solution of inviscid Burger's equation; Dashed line: the spectral approximation without viscosity with $N = 20, 30, 40$ and 60 , respectively. Bold dotted line: the first three are produced by the spectral viscosity method (1.6) with $\epsilon_N = 0.05N^{-0.33}$ and different N s.

N	40	45	50	55	60	65	70
$\ \mathcal{D}_x v_N\ _{L^2(0,T;L^2(\mathbb{R}))}^2$	4.4442	4.5220	4.5099	4.4977	4.6272	4.9252	5.1875
$\ v_N\ _{L^2(0,T;L^2(\mathbb{R}))}^2$	1.8829	1.8824	1.8814	1.8805	1.8804	1.8812	1.8819
$\ x^2 v_N\ _{L^1(\mathbb{R} \times (0,T))}$	1.9499	1.9153	1.8911	1.8671	1.8657	1.8844	1.8791

TABLE 2 $\|\mathcal{D}_x v_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2$, $\|v_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2$ and $\|x^2 v_N\|_{L^1(\mathbb{R} \times (0,T))}$ versus N , respectively, are displayed, where u_N is the solution obtained by the spectral viscosity method (1.6) with $\epsilon_N = 0.05N^{-0.33}$.

Figure 4.3 displays the results of the spectral viscosity method (1.6) with $\epsilon_N = 0.05N^{-0.33}$ and various $N = 20, 30, 40$ and 60 , respectively. It reveals that the larger N is, the better resolution at discontinuity is obtained. It is natural to ask how to tune ϵ_N . With fixed $N = 40$, the results with $\epsilon_N = 0.05N^{-0.45}$, $0.05N^{-0.33}$ and $0.05N^{-0.2}$ are plotted in Figure 4.4. We need to balance the resolution near the discontinuity and the oscillations away from the discontinuity in the choice of ϵ_N . With the same N , the smaller ϵ_N yields the better resolution of the discontinuity, yet the more oscillations away from the discontinuity. In Table 2 we display $\|\mathcal{D}_x v_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2$, $\|v_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2$ and $\|x^2 v_N\|_{L^1(\mathbb{R} \times (0,T))}$ versus N , respectively, where v_N is the numerical solution to (1.6) with $\epsilon_N = 0.05N^{-0.33}$. The norm $\|\cdot\|_{L^2(0,T;L^2(\mathbb{R}))}^2$ is computed using the same rule as before. The L^1 -norm in space in $\|x^2 v_N\|_{L^1(\mathbb{R} \times (0,T))}$ is computed by trapezoid rule using equidistant grid. Again, we obtain the growth rate of $\|\mathcal{D}_x v_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2$, $\|v_N\|_{L^2(0,T;L^2(\mathbb{R}))}^2$ and $\|x^2 u_N\|_{L^1(\mathbb{R} \times (0,T))}$ versus N by using the command “polyfit” in Matlab, which are $N^{0.2431}$, $N^{-0.0014}$ and $N^{-0.0639}$, respectively. The condition (3.36) has been numerically verified.

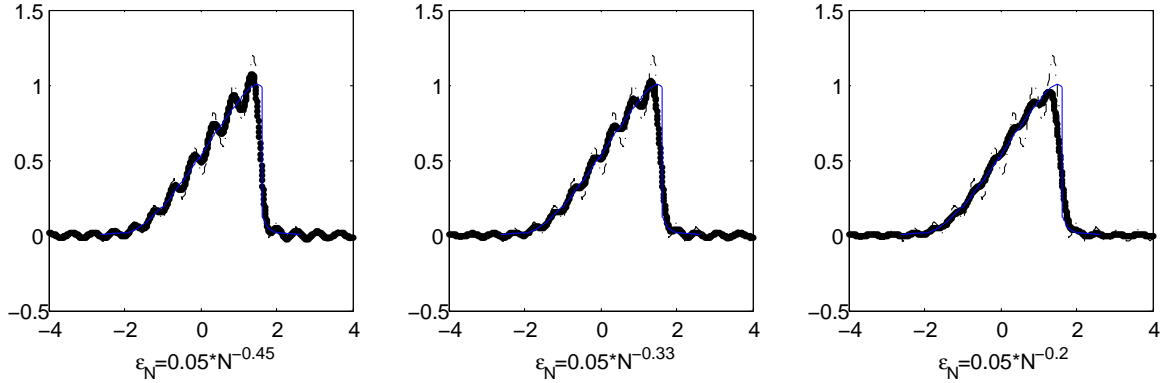


FIGURE 4.4 Solid blue line: the exact solution of inviscid Burger’s equation; Dashed line: the spectral approximation without viscosity with $N = 40$. Bold dotted line: the spectral viscosity method (1.6) with $N = 40$ and different $\epsilon_N = 0.05N^{-0.45}$, $0.05N^{-0.33}$ and $0.05N^{-0.2}$, respectively.

5. CONCLUSION

In this paper, we propose two spectral viscosity methods based on the generalized Hermite functions for the solution of nonlinear scalar conservation laws in the whole line. Our scheme has been shown rigorously to converge to the unique entropy solution by using compensated compactness arguments. The numerical experiments illustrate the implementability of our schemes. Thanks to the generalized Hermite functions, the fewer oscillations and better resolutions compared with those in [1] are obtained with much smaller truncation modes N , even before adding the viscosity term.

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